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LOCALIZATION AT HEIGHT ONE PRIME IDEALS IN AFFINE PRIME PI RINGS

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ABSTRACT

Let R be a prime PI ring that is finitely generated as an algebra over a central subfield. A description of the finite localizable cliques of height one prime ideals of R is given. This description leads to a criterion for the localizability of a height one prime ideal that is identical to the one discovered by Braun and Warfield in the noetherian case.

In [Bra 3], Braun gives a criterion for the localizability of a height one prime ideal P in a prime PI ring R that is finitely generated as a ring over a noetherian central subring A. However, he imposes the severe restriction that P/P^2 be a finitely generated right R-module. Our intention here is to remove this restriction in the important case that R is an affine prime PI k-algebra where k is a field, thus answering Questions 2 and 3 of [Bra 3] in this case. In fact, we prove a more general theorem that describes the clans (or cliques) of height one prime ideals.

Throughout the paper, R will denote an affine prime PI algebra: that is, R is a prime PI ring that is finitely generated as a ring over some central subfield k. Recall that R is contained in its *trace ring* T, a possibly larger subring of the quotient ring of R. The ring T is generated over R by central elements of the quotient ring and T is noetherian when R is affine; in fact T is a finite module over its noetherian centre Z and both T and Z are affine over k [Mc–R, 13.9.11(ii)]. The localizability criterion involves prime ideals of T that lie over P and there are only finitely many of these when P has height one [Bra 3, Lemma 1]. A prime ideal Q of R is said to be *trace-linked* to P if there are prime ideals $\tilde{P}$ and $\tilde{Q}$ of T such that $\tilde{P} \cap R = P$, $\tilde{Q} \cap R = Q$, and

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$\tilde{P} \cap Z = \tilde{Q} \cap Z$. We will denote by Tr($P$) the set of all prime ideals of R that are trace-linked to P. A set of prime ideals X of R is said to be *trace-closed* if $\mathrm{Tr}(P) \subseteq X$ whenever $P \in X$. If $X = X(P)$ is the smallest trace-closed set of prime ideals that contains P, then X is the *trace-closure of* P.

If P is a prime ideal of R, then we say that *lying-over* holds for P if there is a prime ideal $\tilde{P}$ of T such that $P = \tilde{P} \cap R$.

LEMMA 1. *If P is a height one prime ideal of R then*:

(i) *lying over holds for P*,

(ii) *there are only finitely many prime ideals of T that lie over P*,

(iii) *if $\tilde{P}$ lies over P then* height($\tilde{P}$) = 1.

PROOF. (i) [R, Theorems 4.3.7, 4.3.8] and [Mc–R, p. 487 bottom of the page].

(ii) and (iii) [Bra 3, Lemma 1].

LEMMA 2. *Let P be a height one prime of R with trace-closure X. Then* (i) Tr(P) *is finite, and* (ii) *if X consists entirely of height one prime ideals then X is finite*.

PROOF. (i) Let $Q \in \mathrm{Tr}(P)$. Then there are prime ideals $\tilde{P}$ and $\tilde{Q}$ in T such that $\tilde{P} \cap R = P$, $\tilde{Q} \cap R = Q$, and $\tilde{P} \cap Z = \tilde{Q} \cap Z$. Now, by Lemma 1(ii), there are only finitely many choices for $\tilde{P}$. Once $\tilde{P}$ is fixed then there are only finitely many choices for $\tilde{Q}$ by [Bra 2, Proposition 5] and [Bra–S, Lemma 2].

(ii) Choose $0 \neq z$ to be an element of P that is central in R. It is easy to see that if $Q \in X$ then $z \in Q$. Since each such Q has height one, then every Q in X is a minimal prime over the ideal zR. But an affine PI ring has only finitely many minimal prime ideals, by [Mc–R, Corollary 13.4.4]. Thus X is a finite set.

REMARKS. Lying over can fail for prime ideals of height greater than one, as an example in [Sch] shows. An example in [Am–S, p. 381] can be used to show that the restriction on the trace-closure X in Lemma 2(ii) is necessary, for in that example there exists a height one prime ideal P of R such that the trace-closure $X(P)$ is infinite. (The details of this example were worked out with Arthur Chatters.)

Now, let $X = X(P)$ be the trace-closure of a height one prime ideal P and assume that X consists entirely of height one prime ideals and so X is finite by Lemma 2(ii). Set $N = \{\bigcap Q \mid Q \in X\}$. The aim is to show that the semiprime ideal N is localizable; that is, the set $\mathscr{C}(N) = \bigcap \mathscr{C}(Q)$ is an Ore set, where for any ideal I of R the set $\mathscr{C}(I)$ consists of the elements of R that are regular modulo I.

Since the trace ring T is both noetherian and module finite over its centre Z, the localization theory for T is very well-known, see e.g. [Mü], [Bra–S] or [Bra–W]. The idea is to use lying over for height one primes in R to link the problem of localizing in R to the known localization theory in T. To be more specific we need to introduce some notation. Let $\tilde{X} = \{\tilde{Q} \mid \tilde{Q}$ lies over some $Q \in X\}$. Note that $\tilde{X}$ is a finite set of height one prime ideals of T. Also, if $\tilde{Q} \in \tilde{X}$ and A is a prime ideal of T such that $\tilde{Q} \cap Z = A \cap Z$ then $A \in \tilde{X}$, since $A \cap R \in X$ and A lies over $A \cap R$. Set $\tilde{N} = \{\bigcap \tilde{Q} \mid \tilde{Q} \in \tilde{X}\}$. By a result of Müller, [Mü, Theorem 7], the set $\tilde{X}$ of prime ideals of T that are minimal over $\tilde{N}$ form a finite union of clans and so $\tilde{N}$ is a localizable semiprime ideal of T. In fact, this localization is central localization [Mü].

The plan is to exploit the existence of the localization of $\tilde{N}$ in T. Fix the following notation: $\mathscr{S} = \mathscr{C}_R(N)$, $\mathscr{D} = \mathscr{C}_T(\tilde{N})$ and $\mathscr{C}$ the set of central elements for which $T_{\mathscr{D}} = T_{\mathscr{C}}$. Thus, if $p_1, \ldots, p_n$ are the distinct prime ideals in Z such that $\tilde{Q} \cap Z = p_i$, for $\tilde{Q} \in \tilde{X}$, then $\mathscr{C} = Z \setminus (p_1 \cup \cdots \cup p_n)$, by [Mü].

LEMMA 3. $\mathscr{S} \subseteq \mathscr{D}$.

PROOF. It is enough to show that if P, $\tilde{P}$ are height one prime ideals of R, T respectively, such that $\tilde{P} \cap R = P$, then $\mathscr{C}_R(P) \subseteq \mathscr{C}_T(\tilde{P})$. For then,

$$\begin{aligned} \mathscr{S} = \mathscr{C}_R(N) &= \{\bigcap \mathscr{C}(Q) \mid Q \in X\} \\ &\subseteq \{\bigcap \mathscr{C}(\tilde{Q}) \mid \tilde{Q} \in \tilde{X}\} = \mathscr{C}(\tilde{N}). \end{aligned}$$

Let $c \in \mathscr{C}_R(P)$ and let I be the biggest two-sided ideal of R that is contained in $cR + P$. Note that $P \subsetneq I$, by [Mc–R, 13.2.9]. Now T is a central extension of R, so IT is an ideal of T. Suppose that $tc \in \tilde{P}$ for some $t \in T \setminus \tilde{P}$. Then $TtT \cdot IT = TtIT \subseteq TtcT + TtPT \subseteq \tilde{P}$. Since $TtT \not\subseteq \tilde{P}$ we conclude that $IT \subseteq \tilde{P}$. But then $I \subseteq \tilde{P} \cap R = P$, a contradiction. Hence $c \in \mathscr{C}_T(\tilde{P})$.

LEMMA 4. *Let $c \in \mathscr{C}$ and $P \in X$. Then $cT \cap R \not\subseteq P$. In particular, $cT \cap \mathscr{S} \neq \varnothing$.*

PROOF. Let $\tilde{Q}$ be any prime ideal of T that is minimal over c. Then height$(\tilde{Q}) = 1$, by the principal ideal theorem [Mc–R, 4.1.11], since T is noetherian. Now $c \notin \mathscr{C}(\tilde{Q})$, so $\tilde{Q} \notin \tilde{X}$. Thus, all the prime ideals that are minimal over cT lie outside X and so there exist prime ideals $\tilde{Q}_1, \ldots, \tilde{Q}_n$ (not necessarily distinct) in T such that $\Pi_{i=1}^n \tilde{Q}_i \subseteq cT$. Now each $\tilde{Q}_i \cap R$

is a nonzero prime ideal of R that is not contained in X (for otherwise $\tilde{Q}_i$, lying over $\tilde{Q}_i \cap R$, would be in $\tilde{X}$). Hence $0 \neq \Pi_{i=1}^n (\tilde{Q}_i \cap R) \not\subseteq P$, since $P \in X$. But $\Pi_{i=1}^n (\tilde{Q}_i \cap R) \subseteq cT \cap R$, so $cT \cap R \not\subseteq P$. An easy argument shows that $cT \cap \mathcal{S} \neq \emptyset$.

The previous two Lemmas provide the tools for moving between R and T. We need to define symbolic powers for certain ideals in R, but because of the lack of chain conditions in R it is not obvious that the usual type of definition produces a two-sided ideal. We avoid the problem by extending and contracting between R and T.

Let $I \subseteq R$ be an ideal of R *and* T and define symbolic powers, for each $m \geqq 1$, by

$$I^{(m)} = (IT_{\mathcal{C}})^m \cap R.$$

Note that $I^{(m)}$ is a two-sided ideal of R and that

$$I^{(m)} = (T_{\mathcal{C}}I)^m \cap R = I^m T_{\mathcal{C}} \cap R = (IT_{\mathcal{D}})^m \cap R.$$

The next result shows that these symbolic powers behave as one would expect.

LEMMA 5. *Let I be a common ideal of R and T. If $r \in I^{(m)}$ then*
(i) *there exists $s \in \mathcal{S}$ such that $rs \in I^m$, and*
(ii) *there exists $s \in \mathcal{S}$ such that $sr \in I^m$.*

PROOF. (i) Since $r \in I^{(m)}$, there exists $c \in \mathcal{C}$ such that $rc \in I^m$. Thus $rcT \subseteq I^m$, since I is an ideal of T. Now there exists $s \in cT \cap \mathcal{S}$, by Lemma 4, and so $rs \in I^m$, as required. (ii) follows by a symmetric argument.

The conductor C of T into R is a nonzero common ideal of R and T, by [Mc–R, 13.9.6], and so $0 \neq CN$ is an ideal of T that is contained in N (note that $NT = TN$ since T is a central extension of R). Let I be the biggest ideal of T that is contained in N.

LEMMA 6. *If I is the biggest ideal of T that is contained in N then, for each $m \geqq 1$, N is the nilpotent radical of $I^{(m)}$.*

PROOF. If $r \in I^{(m)}$ then there exists $s \in \mathcal{S}$ such that $rs \in I^m \subseteq N$. Thus $r \in N$ and so $I^{(m)} \subseteq N$. It is enough to show that $N^n \subseteq I^{(m)}$, for some n. To achieve this it is sufficient to show that $I^{(m)}$ contains a product of members of X. Now there exist not necessarily distinct prime ideals $P_1, \ldots, P_n$ minimal over $I^{(m)}$ such that $P_1 \cdots P_n \subseteq I^{(m)}$, by [Bra 1]. Suppose that n has been chosen as small as possible. Note that for any ideals A, B of R, $ABT_{\mathcal{C}} = (AT_{\mathcal{C}})(BT_{\mathcal{C}})$, since

T is a central extension of R and $\mathscr{C}$ consists of central elements. Suppose that $P_i \notin X$. Then $P_i \cap \mathscr{S} \neq \varnothing$ and so $P_i \cap \mathscr{D} \neq \varnothing$ by Lemma 3. Hence $P_iT_{\mathscr{C}} = P_iT_{\mathscr{D}} = T_{\mathscr{D}} = T_{\mathscr{C}}$. Thus

$$P_1 \cdots P_{i-1}P_{i+1} \cdots P_nT_{\mathscr{C}} = P_1 \cdots P_nT_{\mathscr{C}} \subseteq (IT_{\mathscr{C}})^m$$

and so

$$P_1 \cdots P_{i-1}P_{i+1} \cdots P_n \subseteq (IT_{\mathscr{C}})^m \cap R = I^{(m)},$$

a contradiction. Hence each $P_i \in X$, as required.

LEMMA 7. *If I is the biggest ideal of T that is contained in N, then $\mathscr{C}(N) \subseteq \mathscr{C}(I^{(m)})$, for each $m \geqq 1$.*

PROOF. Let $s \in \mathscr{S} = \mathscr{C}(N)$ and suppose that $rs \in I^{(m)}$ for some $r \in R$. Then there exists $s_1 \in \mathscr{S}$ such that $r(ss_1) = (rs)s_1 \in I^m$, by Lemma 5. Now $ss_1 \in \mathscr{S} \subseteq \mathscr{D}$, so $r \in I^m(ss_1)^{-1} \subseteq I^mT_{\mathscr{D}}$. Hence $r \in I^mT_{\mathscr{D}} \cap R = I^{(m)}$. A symmetric argument shows that $sr \in I^{(m)}$ implies that $r \in I^{(m)}$, so $\mathscr{C}(N) \subseteq \mathscr{C}(I^{(m)})$.

Experts will recognize the condition $\mathscr{C}(N) \subseteq \mathscr{C}(I^{(m)})$ as the condition appearing in Small's Theorem on the existence of artinian quotient rings. However the rings $R/I^{(m)}$ need not be noetherian and so Small's Theorem is not immediately applicable. We get around this problem by using properties of Gelfand–Kirillov dimension and will use [KL] as a general reference.

COROLLARY 8. *If $a \in R$ and $s \in \mathscr{S}$, then there exist elements $b \in R$, $s_1 \in \mathscr{S}$, and $g \in I^{(m)}$, such that*

$$as_1 - sb = g.$$

PROOF. Set $A = R/I^{(m)}$ and $B = N/I^{(m)}$, so that B is the nilpotent radical of the ring A and $\mathscr{C}(B) \subseteq \mathscr{C}(0)$ by Lemma 6 and Lemma 7. We need to show that $\mathscr{C}(B)$ is a right Ore set in A. Now any affine prime PI algebra has finite, integral Gelfand–Kirillov dimension, by [KL, Corollary 10.6]. Let $\operatorname{GKdim}(R) = n$. If $P \in X$ then height$(P) = 1$, so $\operatorname{GKdim}(R/P) = n - 1$, by Schelter's Catenarity Theorem [Sch] and [KL, Theorem 10.10]. Since $I^{(m)}$ is a nonzero ideal of the prime PI ring R, $\operatorname{GKdim}(R/I^{(m)}) \leqq n - 1$, by [KL, Proposition 3.15]. Hence

$$n - 1 \geqq \operatorname{GKdim}(A) = \operatorname{GKdim}(R/I^{(m)}) \geqq \operatorname{GKdim}(R/N) \geqq \operatorname{GKdim}(R/P) = n - 1.$$

Thus $\operatorname{GKdim}(A) = \operatorname{GKdim}(A/B) = n - 1$, and if P is a minimal prime ideal of A then $\operatorname{GKdim}(A/P) = n - 1$ also.

Let $a \in A$ and $c \in \mathscr{C}(B) \subseteq \mathscr{C}(0)$. Set $E = \{e \in A \mid ae \in cA\}$. Then left multiplication by a produces an embedding of A/E into A/cA. Thus $\mathrm{GKdim}(A/E) \leqq \mathrm{GKdim}(A/cA) < n-1$, since $c \in \mathscr{C}(0)$. If P is a minimal prime ideal of A then

$$\mathrm{GKdim}(A/E+P) \leqq \mathrm{GKdim}(A/E) < n-1 = \mathrm{GKdim}(A/P).$$

It follows that $E+P/P$ is an essential right ideal of A/P, by [KL, Lemma 5.12 and Lemma 5.13]. Thus $(E+P) \cap \mathscr{C}(P) \neq \varnothing$ and so $E \cap \mathscr{C}(P) \neq \varnothing$. An easy argument shows that $E \cap \mathscr{C}(B) \neq \varnothing$, and the result follows.

Remark. This is the only place where we use the condition that R is affine over a field. Everything else works for R affine over a central noetherian subring.

In our attempt to localize at $\mathscr{C}(N)$ we have now shown that this is possible modulo $I^{(m)}$, for each $m \geqq 1$. The usual next step is to use an Artin–Rees type argument. In order to do this, we need to move back up to the localization of the trace ring.

Proposition 9. *If I is the biggest ideal of T that is contained in N, then $I_{\mathscr{C}}$ is an ideal of $T_{\mathscr{C}}$ that has the Artin–Rees property.*

Proof. Let J be the Jacobson radical of the semilocal ring $T_{\mathscr{C}}$. Then $J = \tilde{N}_{\mathscr{C}}$ and so $I_{\mathscr{C}} \subseteq J$, since $I \subseteq N \subseteq \tilde{N}$. Now the maximal ideals of $T_{\mathscr{C}}$ are induced from the members of $\tilde{X}$ and these are all height one prime ideals. Hence $T_{\mathscr{C}}$ is a semilocal noetherian prime PI ring with Krull dimension one. Thus $T_{\mathscr{C}}/I_{\mathscr{C}}$ is artinian and so $J^m \subseteq I_{\mathscr{C}}$, for some m; in a similar manner one sees that J has the Artin–Rees property. Let E be any right ideal of $T_{\mathscr{C}}$. Then there exists n such that $E \cap J^n \subseteq EJ^m$. Hence $E \cap I^n_{\mathscr{C}} \subseteq E \cap J^n \subseteq EJ^m \subseteq EI_{\mathscr{C}}$.

Corollary 10. *N is a localizable semiprime ideal of R.*

Proof. Let $a \in R$ and $s \in \mathscr{S} = \mathscr{C}(N)$. Set $E = aR + sR$. There exists an integer $m \geqq 1$ such that $E \cap I^m_{\mathscr{C}} \subseteq ET_{\mathscr{C}} \cap I^m_{\mathscr{C}} \subseteq ET_{\mathscr{C}} I_{\mathscr{C}} = EI_{\mathscr{C}}$, by Proposition 9. Now there exist $d \in \mathscr{S}$, $b \in R$ and $g \in I^{(m)}$ such that $ad - sb = g$, by Lemma 8. Note that

$$g = ad - sb \in E \cap I^{(m)} \subseteq E \cap I^m_{\mathscr{C}} \subseteq EI_{\mathscr{C}} = aI_{\mathscr{C}} + sI_{\mathscr{C}}.$$

Set $g = ax + sy$, for some $x, y \in I_{\mathscr{C}}$ and choose $c \in \mathscr{C}$ such that $sc, yc \in I$. Then $xcT, ycT \subseteq I$, since I is an ideal of T. Now there exists an element $s_1 \in cT \cap \mathscr{S}$, by Lemma 4. Note that $xs_1 \in xcT \subseteq I$ and similarly $ys_1 \in I$. Thus $ads_1 - sbs_1 =$

$gs_1 = axs_1 + sys_1$ and so $a(ds_1 - xs_1) = s(bs_1 + ys_1)$. Although x, y need not be in R the elements xs_1 and ys_1 are in I and so in R. Set

$$s_2 = ds_1 - xs_1 \in R \quad \text{and} \quad b_1 = bs_1 + ys_1 \in R$$

and note that $ds_1 \in \mathcal{S}$ and $xs_1 \in I \subseteq N$, so $s_2 \in \mathcal{S}$. Thus the equation $as_2 = sb_1$ verifies the Ore condition for $\mathcal{S}$ in R.

In summary, we have

THEOREM 11. *Let R be an affine prime PI algebra and let P be a height one prime ideal of R. If the trace-closure $X = X(P)$ consists entirely of height one prime ideals of R, then $\{\bigcap \mathscr{C}(Q) \mid Q \in X\}$ is an Ore set.*

Looking at the special case where $\mathrm{Tr}(P) = \{P\}$, we have

THEOREM 12. *Let R be an affine prime PI algebra and let P be a height one prime ideal of R. Then the following are equivalent:*
(i) $\mathrm{Tr}(P) = \{P\}$,
(ii) *P is right localizable,*
(iii) *P is left localizable.*

PROOF. If $\mathrm{Tr}(P) = \{P\}$ then $X(P) = \{P\}$, so Theorem 11 gives (i)$\Rightarrow$(ii), (iii).

(ii)$\Rightarrow$(i). This is essentially proved in [Bra–W], but although they assume R noetherian, this condition is not needed at height one. Let P be right localizable and suppose that $Q \in \mathrm{Tr}(P)$. Thus there exist prime ideals $\tilde{P}$, $\tilde{Q}$ in T such that $P = \tilde{P} \cap R$, $Q = \tilde{Q} \cap R$ and $\tilde{P} \cap Z = \tilde{Q} \cap Z$. Since T is centrally generated over R and $\mathscr{C}(P)$ is a right Ore set in R, it follows that $\mathscr{C}(P)$ is a right Ore set in T. Now, certainly,

$$\mathscr{C}(P) \cap \tilde{P} \subseteq \mathscr{C}(P) \cap \tilde{P} \cap R = \mathscr{C}(P) \cap P = \varnothing,$$

so $\mathscr{C}(P) \subseteq \mathscr{C}(\tilde{P})$. Now $\tilde{P}$ and $\tilde{Q}$ belong to the same clique, by [Mü, Theorem 7] or [Bra–W, Proposition 3], so $\mathscr{C}(P) \subseteq \mathscr{C}(\tilde{Q})$. If $Q \not\subseteq P$ then $Q \cap \mathscr{C}(P) \neq \varnothing$, thus $Q \subseteq P$. Since height$(P) = 1$, this forces $Q = P$. Hence $\mathrm{Tr}(P) = \{P\}$.

(iii)$\Rightarrow$(i) follows in a similar way.

There seems to be little chance of generalising these ideas to primes of height greater than one. Indeed, lying over may fail and so trace-linkage makes little sense. Some progress might be possible for rings of generic matrices, for in this case lying over does hold [Am–S, Theorem 4.3].

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